Self-similarity in Newtonian Cosmology

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Motivation

- The properties and existence of the dark matter is one of the most fascinating questions in cosmology.
- We present a **dark fluid model** described as a non-relativistic and self-gravitating fluid.
- We studied these coupled **non-linear differential equation** systems using self-similar time-dependent solutions
- Our main goal of this research is to find **scaling solutions** of the gravitational fields, which can be good candidates to describe the evolution of the Universe or collapse of compact astrophysical objects.

The Model

- Continuity Equation

$$\partial_t \rho + \mathrm{div}(\rho \boldsymbol{u}) = 0,$$

$$\partial_t \rho \boldsymbol{u} + \mathrm{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) = -\nabla P(\rho) - \rho \nabla \Phi + \rho \boldsymbol{g}$$

$$\nabla^2 \Phi = 4\pi G \rho$$

$$P = P(\rho)$$

- Continuity Equation
- Euler Equation

$$\partial_t \rho + \mathrm{div}(\rho \boldsymbol{u}) = 0,$$

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- Poisson Equation

- Continuity Equation
- Euler Equation

$$\partial_t \rho + \operatorname{div}(\rho \boldsymbol{u}) = 0,$$

$$\partial_t \rho \boldsymbol{u} + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) = -\nabla P(\rho) - \rho \nabla \Phi + \rho \boldsymbol{g}$$

$$\nabla^2 \Phi = 4\pi G \rho$$

$$P = P(\rho)$$

- Poisson Equation
- Equation of State

- We used polytropic EoS:

$$P(\rho) = w\rho^n, \qquad \text{where } n = 1$$

- Dark Fluid: $w = -1$
- Momentum conservation:

$$\nabla P(\rho) + \rho \nabla \Phi = 0$$

- Rotation:

$$\rho \mathbf{g} = \frac{\rho \sin\theta \omega^2 r}{t^2} \quad \omega : \text{angular velocity}$$

- Rotation is slow! $\Rightarrow$ Asymptotic spherically symmetry
- Spherical Symmetry:

$$\partial_t \rho + (\partial_r \rho) u + (\partial_r u) \rho + \frac{2u\rho}{r} = 0,$$

$$\partial_t u + (u \partial_r) u = -\frac{1}{\rho} \partial_r P - \nabla \Phi + \frac{\sin\theta \omega^2 r}{t^2},$$

$$\Delta \Phi = 4\pi G \rho \ .$$

$$P = P(\rho) \ .$$

- Self-similarity in 1D ⇒ Sedov – Taylor *ansatz*
  G. I. Taylor, British Report RC-210, June 27, (1941)
  IF Barna, MA Pocsai, GG Barnaföldi Mathematics 10 (18), 3220 (2022)

$$u(r,t) = t^{-\alpha} f\left(\frac{r}{t^{\beta}}\right) \quad \rho(r,t) = t^{-\gamma} g\left(\frac{r}{t^{\beta}}\right)$$

$$\Phi(r,t) = t^{-\delta} h\left(\frac{r}{t^{\beta}}\right),$$

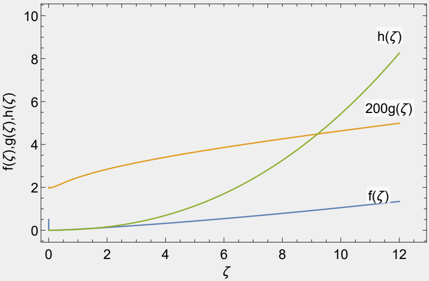
- (f, g, h) shape-functions only depend on $\zeta = rt^{-\beta}$
- Similarity exponents: $\alpha, \beta, \gamma, \delta$
- The β describes the rate of spread of the spatial distribution
- Other exponents describe the rate of decay of the intensity of the corresponding field

- Self-Similarity: PDE reduce to ODE
- Depend only on ζ self-similar variable
- Algebraic equation system for the exponents
  $\Rightarrow \alpha = 0,\ \beta = 1,\ \gamma = 2,$ and $\delta = 0$

$$-\zeta g'(\zeta) + f'(\zeta)g(\zeta) + f(\zeta)g'(\zeta) + \frac{2f(\zeta)g(\zeta)}{\zeta} = 0,$$

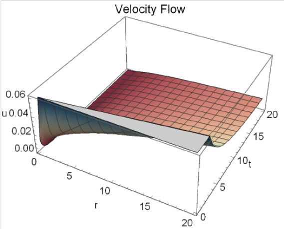
$$-\zeta^2 f'(\zeta) + \zeta f'(\zeta)f(\zeta) + \frac{wg'(\zeta)}{g(\zeta)} = -h'(\zeta)\zeta + \omega^2 \sin\theta\zeta^2,$$

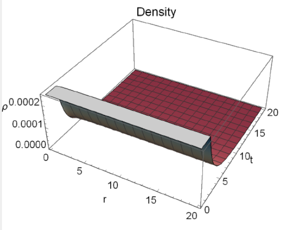
$$h'(\zeta) + h''(\zeta)\zeta = g(\zeta)4\pi G\zeta\ .$$

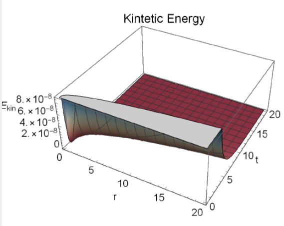


- $\zeta_0 = 0.0001$
- $f(\zeta_0) = 0.5, g(\zeta_0) = 0.001$
- $f(\zeta_0)$ is linear
- $g(\zeta_0)$ is polynomial with a < 1 exponent
- $h(\zeta_0)$ is polynomial with a > 1 exponent

- **Kinetic Energy:** $\epsilon_{kin}(r,t) = \frac{1}{2}\rho(r,t)u^2(r,t)$



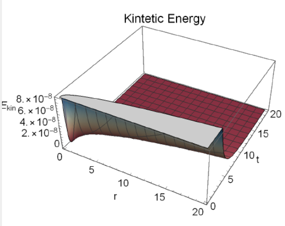


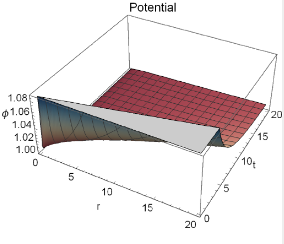


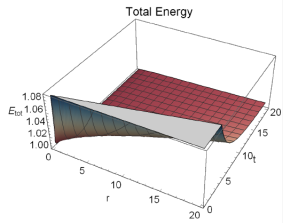
- Distributions have a singularity at the origin
- Solutions decay over time
- Radial profile shows different behaviour

Dynamical Variables II.

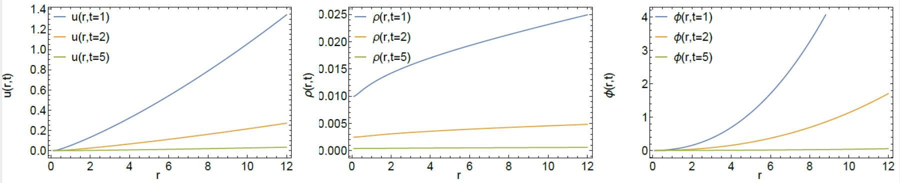
- **Total Energy:** $\epsilon_{tot}(r,t) = \epsilon_{kin}(r,t) + \Phi(r,t)$



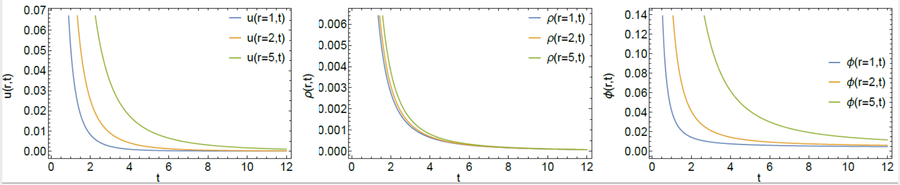




- Distributions have a singularity at the origin
- They decay over time



- Non-rotating case
- density increases excessively near the center of the explosion distances and becomes linear at large distances
- Velocity grew polynomially with the radial distance.
- Gravitational potential grows polinomically



- Non-rotating case
- The domain range is given in geometrized unit.
- They decrease hyperbolically over time, and become asymptotically flat.
- They have a real singularity at $t = 0$, due to the shape of the *ansatz*.

Connection to Friedmann Equation

Newtonian Friedmann Equation

- We are introducing a well-known scale-factor $a(t)$ which contains all of the temporal changes
- Relative distances in time: $R(t) = a(t)l$
- $\Omega(t) \subset \mathbb{R}^3$ is a sphere with radius $R(t)$ and $r \in (0, R(t))$

Mass

$$M(t) = \int_{\Omega(t)} \rho(R(t), t) dV = 4\pi \int_0^r \rho(R(t), t) R(t)^2 dR(t)$$

Mass Conservation

$$\frac{d}{dt} M(t) = 4\pi \frac{d}{dt} \int \rho(a(t)l, t) a^3(t) l^2 dl \stackrel{!}{=} 0$$

First Friedmann Equation

$$\frac{\frac{d}{dt}[\rho(a(t)l, t)]}{\rho(a(t)l, t)} = -3\frac{\dot{a}(t)}{a(t)}$$

Kinematic Condition

$$\frac{d}{dt}R(t) = u(R(t), t) \Rightarrow \frac{\frac{d}{dt}\left[t^{-\gamma}g(R(t), t)\right]}{g(R(t), t)} = -3\frac{t^{-\alpha}f(R(t), t)}{R(t)}$$

Power series in the similarity variable

$\rho(r,t) \sim t^{-\gamma} \sum_n^{\infty} \rho_n \zeta^n$ and $u(r,t) \sim t^{-\alpha} \sum_n^{\infty} u_n \zeta^n$

In the relevant space and time scale

- $\rho(r,t) \sim t^{-\gamma} A \zeta^{\kappa}$, where $\kappa \in \mathbb{R}^+$

We assume, that

$\rho(r,t) \sim t^{-\gamma} A \zeta^{\kappa}$, and $u(r,t) \sim t^{-\alpha} \sum_n^{8} u_n \zeta^n$

Non-rotating case: $\omega \to 0$ limit

Non-rotating:

$$u(r,t) \sim t^{-\alpha} \left(u_1 \zeta^1 + u_2 \zeta^2 \right)$$

Summarizing this,

Non-Rotating:	**Rotating:**
$\rho(r,t) \sim t^{-\gamma} A \zeta^{\kappa}$	$\rho(r,t) \sim t^{-\gamma} A \zeta^{\kappa}$
$u(r,t) \sim u_1 \zeta + u_2 \zeta^2$	$u(r,t) \sim t^{-\alpha} \sum_{k=0}^{8} \tilde{u}_k \zeta^k$

- Non-autonomous first-order non-linear differential equation

$$\kappa \dot{R}(t) + 3u_2 t^{-(\alpha+2\beta)}[R(t)]^2 - \frac{1}{t}[\gamma + \kappa\beta]R(t) + 3u_1 R(t) t^{-(\alpha+\beta)} = 0$$

For the non-rotating case, the differential equation is

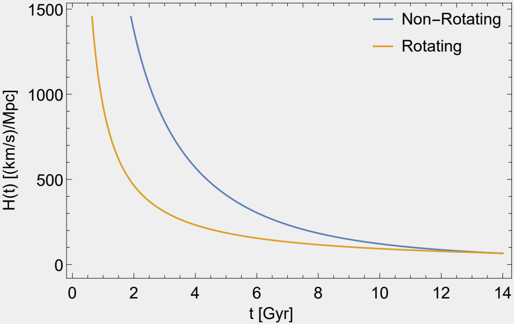
$$\kappa \dot{R}(t) - \frac{1}{t}[\gamma + \kappa\beta]R(t) + 3t^{-\alpha}\sum_{k=0}^{8}\tilde{u}_k\left(\frac{R(t)}{t^{\beta}}\right)^k = 0. \tag{1}$$

- It cannot be solved explicitly
- Hubble's law of expansion to determine the C_1 integration constant

$$\left.\frac{\dot{a}(t)}{a(t)}\right|_{t=t_0} = H_0, \text{ if } \quad a(t_0) = 1 \tag{2}$$

where $H_0 = 66.6^{+4.1}_{-3.3}$ km/s/Mpc[1] is the experimental value of the Hubble-constant.

[1]Kelly, P. L. et al. (2023) Science doi:10.1126/science.abh1322



- Analytical (Non-Rotating) and numerical (Rotating) solutions
- Integration started at $\zeta_0 = 0.001$.
- Initial Cond. $f(\zeta_0) = 0.5$, $g(\zeta_0) = 0.008$, $h(\zeta_0) = 0$
- Shows similarities with literature[1].

[1] Xiaoyun Li, et al. J.HEP, Gravitation and Cosmology, Vol.8 No.1, 2022

Summary

- We used Sedov-Taylor-von Neumann ansatz to solve the Euler-Poisson equation
- We used polytropic EoS to describe the Dark Fluid
- Spherical symmetry and non-rotating/slow rotation
- Connection with the classical Newtonian Friedmann equation
- Expansion rate of the Universe

Thank you!

Questions?

General Solution for non-rotating case:

$$R(t) = \frac{u_1 t^{\beta+\gamma/\kappa} e^{-\frac{3u_1 t^{\mu}}{\mu\kappa}}}{3^{-\frac{\gamma}{\mu\kappa}} u_2 t^{\gamma/\kappa} \left(\frac{u_1 t^{\mu}}{\nu}\right)^{-\frac{\gamma}{\mu\kappa}} \Gamma\left(\frac{\gamma-\beta\kappa+\kappa}{\nu}, \frac{3u_1 t^{\mu}}{\nu}\right) - \mathcal{C}_1 u_1}$$

$$\mu := 1 - (\alpha + \beta) \qquad \nu := \kappa - \beta\kappa$$

- The $\mathcal{C}_1$ is an integration constant
- Γ is the upper incomplete Gamma function.
- $(\alpha, \beta, \gamma, \delta)$ are known from the Sedov-Taylor Ansatz

$$R(t) = \frac{t}{\mathcal{C}_1 t^{\frac{3u_1-2}{\kappa}} + \frac{3u_2}{2-3u_1}}, \quad \text{where } \kappa = \frac{6}{7}$$

Energy Conservation

$$H^2(t) = \frac{8\pi G}{3}\rho(R(t), t) + U_t,$$

where, $U_t = U_0/R(t)^2$ is a dynamical constant.

Entropy Conservation

$$\dot{E} + p\dot{V} = 0 \quad \Rightarrow \quad \frac{\dot{\rho}}{\rho} = -3\left[\frac{\dot{R}}{\dot{R}} + \frac{\ddot{R}}{\dot{R}}\right], \tag{3}$$

$$H(t) = H_0^2\sqrt{\Omega_{0,CDM}\left(\frac{H_0}{H(t)}\right)^3\frac{1}{a^3(t)} + \Omega_{DE,0}\frac{1}{a^2(t)}}.$$